CONTINUOUS CLUSTER CATEGORIES

LECTURES BY GORDANA TODOROV
NOTES TAKEN BY SHIJIE ZHU

Notes taken by Shijie Zhu (Northeastern University) during the Workshop on Cluster Categories at University of Connecticut, May 2017; with some modifications after the Workshop on representation theory of algebras, Tsinghua University, July 4-5, 2017.

Contents

1. Cluster categories (acyclic)

Let $Q = (Q_0, Q_1, s, t)$ be a finite acyclic quiver, where Q_0 is the set of vertices satisfying $|Q_0| < \infty$, and Q_1 is the set of arrows, and s and t indicate the start vertex and target vertex of an arrow (respectively), i.e. we denote $s(\alpha) \xrightarrow{\alpha} t(\alpha)$ for all $\alpha \in Q_1$.

1.1. **Representations of quivers,** $\text{rep}_{\mathbf{k}}(Q)$**.** We denote by $\text{rep}_{\mathbf{k}}(Q)$ the category of finite dimensional representations of the quiver Q over $\mathbf{k}$. The objects of the category $\text{rep}_{\mathbf{k}}(Q)$ are:

$$V = (\{V_a\}_{a \in Q_0}, \{V_\alpha\}_{\alpha \in Q_1}),$$

where $\{V_a\}_{a\in Q_0}$ are finite dimensional $\mathbf{k}$-vector spaces and $\{V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}\}_{\alpha\in Q_1}$ are $\mathbf{k}$-linear transformations.

The morphisms in $\text{rep}_{\mathbf{k}}(Q)$ are given by $f = (f_a)_{a\in Q_0} : V \to W$, such that the following diagrams are commutative:

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{V_\alpha} & V_{t(\alpha)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ W_{s(\alpha)} & \xrightarrow{W_\alpha} & W_{t(\alpha)} \end{array}$$

Let $f = (f_a)_{a\in Q_0} : V \to W$ and $g = (g_a)_{a\in Q_0} : U \to V$ be morphisms in $\text{rep}_{\mathbf{k}}(Q)$, then the composition $f \circ g$ is defined by $(f \circ g)_a = f_a \circ g_a$.

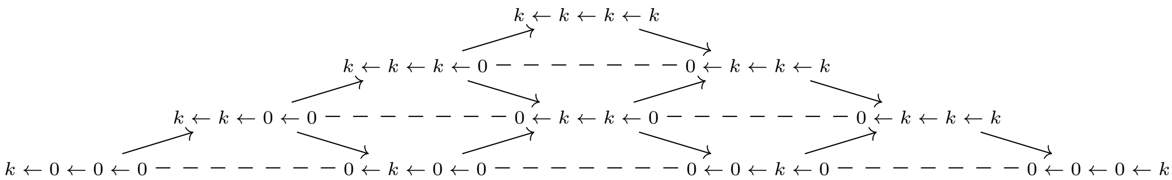
Note that $\text{rep}_{\mathbf{k}}(Q)$ is an abelian category. The category $\text{rep}_{\mathbf{k}}(Q)$ is equivalent to the category of finitely generated modules of the path algebra $\mathbf{k}Q$, i.e. $\text{rep}_{\mathbf{k}}(Q) \cong \text{mod}(\mathbf{k}Q)$.

The Auslander-Reiten quiver (AR-quiver) is a nice way to organize indecomposable modules, irreducible maps and almost split sequences (AR-sequences) of the module category.

Example 1.1.1. Let Q be a quiver of type $\mathbb{A}_4$:

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

Then the AR-quiver of the category $\text{rep}_{\mathbf{k}}(Q)$ looks like:



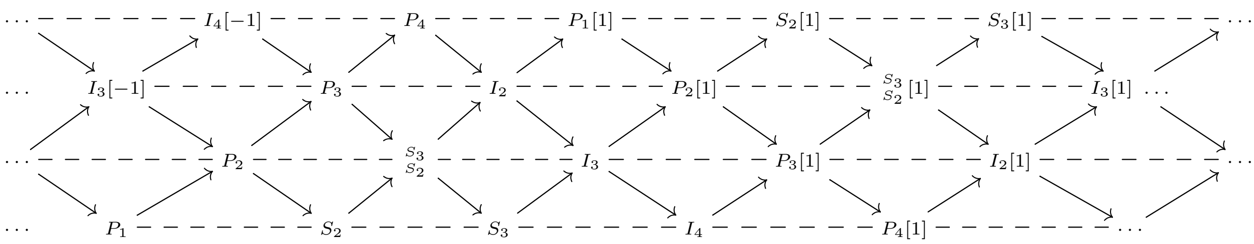
Remark 1.1.2. The category $\text{rep}_{\mathbf{k}}(Q)$ is a hereditary category, i.e.
- the submodules of projectives are projective,
- or the quotients of injectives are injective,
- or $\text{Ext}^i(X, Y) = 0$ for $\forall i \geq 2$.

1.2. **The derived category of $\mathbf{k}Q$, $D^b(\mathbf{k}Q)$.** Since $\text{rep}_{\mathbf{k}}(Q)$ is hereditary, all indecomposable objects in $D^b(\mathbf{k}Q)$ are isomorphic to shifts of indecomposable modules. The AR-quiver of $D^b(\mathbf{k}Q)$ looks like:

$$\cdots \; \boxed{\text{rep}(\mathbf{k}Q)[-1]} \; \boxed{\text{rep}(\mathbf{k}Q)} \; \boxed{\text{rep}(\mathbf{k}Q)[1]} \; \cdots$$

If X and Y are $\mathbf{k}Q$-modules, then
- $\text{Hom}_{D^b(\mathbf{k}Q)}(X, Y) \cong \text{Hom}_{\mathbf{k}Q}(X, Y)$.
- $\text{Hom}_{D^b(\mathbf{k}Q)}(X, Y[1]) \cong \text{Ext}^1_{\mathbf{k}Q}(X, Y)$.
- $\text{Hom}_{D^b(\mathbf{k}Q)}(X, Y[n]) = 0$ for all $\forall n > 1$ or $n < 0$.

Example 1.2.1. Let Q be the same as Example 1.1.1. The AR quiver of $D^b(\mathbf{k}Q)$ looks like:



Let $A \to B \to C \to A[1]$ be an AR triangle, then $C \cong \tau^- A$ where τ^- is the inverse Auslander-Reiten translation. In mod($\mathbf{k}Q$), $\tau^- = TrD$ is the transpose of the dual. Notice:
- τ^- is a functor $D^b(\mathbf{k}Q) \to D^b(\mathbf{k}Q)$.
- The shift [1] is a functor $D^b(\mathbf{k}Q) \to D^b(\mathbf{k}Q)$.
- So $F = \tau^-[1]$ is also a functor $D^b(\mathbf{k}Q) \to D^b(\mathbf{k}Q)$.

1.3. **The cluster category $\mathcal{C}_Q$ of a quiver Q.** [BMRRT] The *cluster category* $\mathcal{C}_Q$ is the orbit category of $D^b(\mathbf{k}Q)$ by the functor $F = \tau^-[1]$. The objects in $\mathcal{C}_Q$ are given by $\{\tilde{X}\}$, which are the orbits of objects X in $D^b(\mathbf{k}Q)$. The morphisms in $\mathcal{C}_Q$ are defined as:

$$\mathrm{Hom}_{\mathcal{C}_Q}(\tilde{X}, \tilde{Y}) = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{D^b(\mathbf{k}Q)}(X, F^i Y),$$

where X, Y are representatives of $\tilde{X}$ and $\tilde{Y}$ respectively.

Notice that $\coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{D^b(\mathbf{k}Q)}(X, F^i Y)$ is a finite sum. In fact, if the representatives X and Y are $\mathbf{k}Q$-modules, then $\mathrm{Hom}_{D^b(\mathbf{k}Q)}(X, F^i Y) = 0$ for all $i \neq 0, 1$.

Theorem 1.3.1 (B.Keller)**.** *The cluster category $\mathcal{C}_Q$ is a triangulated category.*

2. Continuous cluster categories I

This work is done and published in [IT15], [IT11].

2.1. **Representations of the real line $\mathbb{R}$ over k, denoted by $\mathrm{rep}_{\mathbf{k}}\,\mathbb{R}$.** The category $\mathrm{rep}_{\mathbf{k}}\,\mathbb{R}$ will consist of locally finite dimensional representations of $\mathbb{R}$ over $\mathbf{k}$. These representations will be similar to the representations of the quiver A_n with linear orientation. Recall that a typical indecomposable representation of an $\mathbb{A}_n$ type quiver Q with linear orientation $V = (\{V_i\}, V_{t'} \xleftarrow{V_\alpha} V_t)$ looks like:

$$0 \leftarrow k \leftarrow k \leftarrow k \leftarrow 0 \leftarrow 0.$$

Typical (non-zero) morphisms between $\mathbf{k}Q$ looks like:

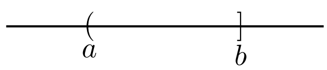
$$\begin{array}{ccccccccccc} 0 & \longleftarrow & k & \longleftarrow & k & \longleftarrow & k & \longleftarrow & 0 & \longleftarrow & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \longleftarrow & 0 & \longleftarrow & k & \longleftarrow & k & \longleftarrow & k & \longleftarrow & 0 \end{array}$$

As a natural generalization, we can define the indecomposable representations of $\mathbb{R}$.

Definition 2.1.1. Let $-\infty \leq a < b < \infty$ be real numbers, define the indecomposable representation

$$V_{(a,b]} := \left(V_{(a,b]}(t) = \begin{cases} \mathbf{k} & a < t \leq b \\ 0 & else \end{cases}, V_{(a,b]}(t') \xleftarrow{=} V_{(a,b]}(t) \right).$$

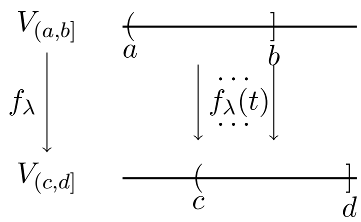
Graphically the representation $V_{(a,b]}$ looks like an open interval:



Morphisms $f_\lambda : V_{(a,b]} \to V_{(c,d]}$ are given by: let $x \in V_{(a,b]}(t)$. Then

$$f_\lambda(x) = \begin{cases} \lambda x & c < t \leq b \\ 0 & else. \end{cases}$$

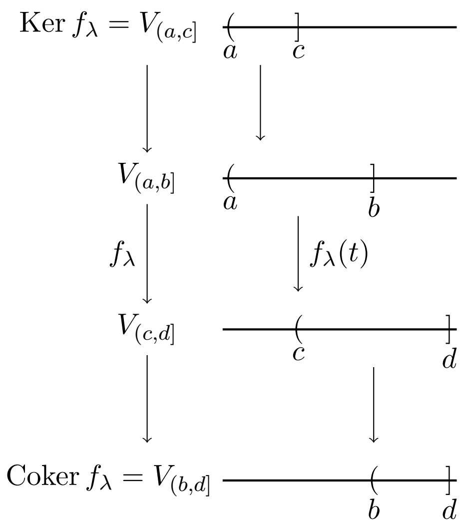
Graphically,



The composition of morphisms is defined point-wise.

Denote by $\text{rep}_{\mathbf{k}}(\mathbb{R})$ the category of (locally finite dimensional) **k**-representations of the real line $\mathbb{R}$, whose objects are finite sums of $\{V_{(a,b]}\}$, $(-\infty \le a < b < \infty)$ and morphisms are linear sums of $\{f_\lambda\}$. Notice that any object in $\text{rep}_{\mathbf{k}}(Q)$ is of finite dimension over **k** at each point.

Remark 2.1.2. • The category $\text{rep}_{\mathbf{k}}(\mathbb{R})$ is an abelian category. For example, let $a < c < b < d$ be real numbers, then the non-zero morphism $f_\lambda : V_{(a,b]} \to V_{(c,d]}$ has kernel and cokernel as:



• $\text{rep}_{\mathbf{k}}(\mathbb{R})$ has indecomposable projectives $P_b = V_{(-\infty,b]}$.

• $\text{rep}_{\mathbf{k}}(\mathbb{R})$ has enough projective objects, i.e. for each object $V_{(a,b]}$, there is an epimorphism $P_b \to V_{(a,b]}$.

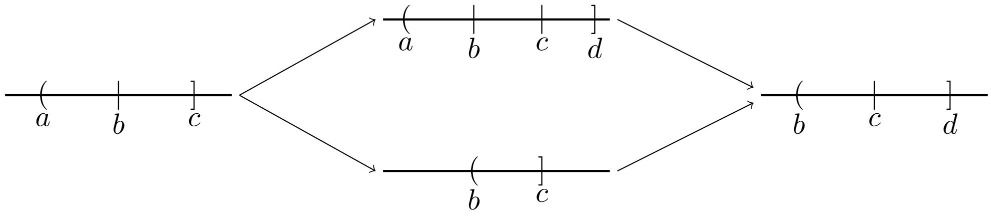
• $\text{rep}_{\mathbf{k}}(\mathbb{R})$ does not have injectives.

Example 2.1.3. We show two examples of exact sequences in $\text{rep}_{\mathbf{k}}(\mathbb{R})$:

(1) $0 \longrightarrow V_{(a,c]} \longrightarrow V_{(a,d]} \longrightarrow V_{(c,d]} \longrightarrow 0$ for $a < c < d$.

(2) $0 \longrightarrow V_{(a,c]} \longrightarrow V_{(a,d]} \oplus V_{(b,c]} \longrightarrow V_{(b,d]} \longrightarrow 0$ for $a < b < c < d$.

Graphically,



Usually we identify the category $\operatorname{rep}_{\mathbf{k}}\mathbb{R}$ with the category of finitely presented objects $\operatorname{pres}_{\mathbf{k}}\mathbb{R}$ defined in the following way:

The indecomposable objects in $\operatorname{pres}_{\mathbf{k}}\mathbb{R}$ are either monomorphisms $P_a \to P_b$, for $a < b$ or $0 \to P_b$. The objects of $\operatorname{pres}_{\mathbf{k}}\mathbb{R}$ are finite sums of indecomposable objects and the morphisms are commutative diagrams.

We can show an equivalence of categories $\operatorname{pres}_{\mathbf{k}}\mathbb{R} \cong \operatorname{rep}_{\mathbf{k}}\mathbb{R}$, through the correspondence:

$$\{P_a \to P_b \mid a < b\} \leftrightarrow V_{(a,b]}$$

$$\{0 \to P_b\} \leftrightarrow P_b.$$

2.2. **Special subcategory of $\operatorname{rep}_{\mathbf{k}}\mathbb{R}$, denoted by $\mathcal{A}_{\mathbb{R}}$.** Let $\mathcal{A}_{\mathbb{R}}$ be the full subcategory of $\operatorname{rep}_{\mathbf{k}}\mathbb{R} \cong \operatorname{pres}_{\mathbf{k}}\mathbb{R}$ closed under direct sums of objects $V_{(a,b]}$ in $\operatorname{rep}_{\mathbf{k}}\mathbb{R}$ with $\infty < a < 0 < b$ or equivalently $\{P_a \to P_b\} \in \operatorname{pres}_{\mathbf{k}}\mathbb{R}$ with $a < 0 < b$.

Remark 2.2.1. • The subcategory $\mathcal{A}_{\mathbb{R}}$ is not an abelian category. Consider the following (non-zero) morphism $f : V_{(-3,2]} \to V_{(-3,7]}$. It is easy to check that f is a monomorphism in $\mathcal{A}_{\mathbb{R}}$, so $\operatorname{Ker} f$ is the zero morphism $g : 0 \to V_{(-3,2]}$. It is also easy to check that f is an epimorphism in $\mathcal{A}_{\mathbb{R}}$, so $\operatorname{Coker} f$ is a zero morphism $h : V_{(-3,7]} \to 0$. But then $\operatorname{Coker} g \ncong \operatorname{Ker} h$.
• In fact, for any non-zero map $f : V_{(a,b]} \to V_{(c,d]}$ in $\mathcal{A}_{\mathbb{R}}$ we have $\operatorname{Coker}\operatorname{Ker} f \neq \operatorname{Ker}\operatorname{Coker} f$.

2.3. **The map $\mathcal{A}_{\mathbb{R}} \to \mathbb{R}^2$.** This will be a map from indecomposable objects to the points in $\mathbb{R}^2$. (This is similar to the map of indecomposable modules to the points of AR-quiver.)

Define the map $\Psi : \mathcal{A}_{\mathbb{R}} \to \mathbb{R}^2$ on indecomposable objects by:

$$\Psi(V_{(a,b]}) = (-\ln(-a), \ln b) \in \mathbb{R}^2.$$

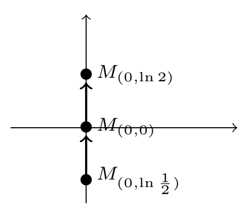
It is easy to see Ψ is an onto.

2.4. $\mathbb{R}^2$ **as an additive category.** We want to describe $\mathbb{R}^2$ as a category with indecomposable objects being the points (x, y). To emphasize that each point (x, y) is considered as an object, we usually denote it by $M_{(x,y)}$. Morphisms are given by

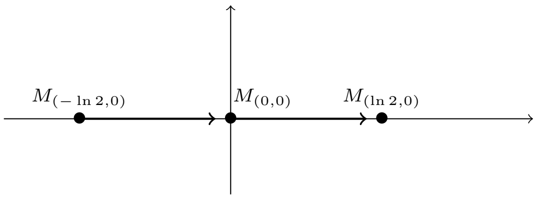
$$\operatorname{Hom}_{\mathbb{R}^2}(M_{(x,y)}, M_{(x',y')}) = \begin{cases} k & x \le x', y \le y' \\ 0 & else. \end{cases}$$

In fact, Ψ is a functor $\mathcal{A}_{\mathbb{R}} \to \mathbb{R}^2$. Using Ψ, we can view the morphisms in $\mathcal{A}_{\mathbb{R}}$ between indecomposable objects as the lines between the corresponding points on the plane $\mathbb{R}^2$. For example: monomorphisms $V_{(a,b]} \to V_{(a,c]}$ for $b \le c$ are vertical and epimorphisms $V_{(a,c]} \to V_{(b,c]}$ for $a \le b$ are horizontal.

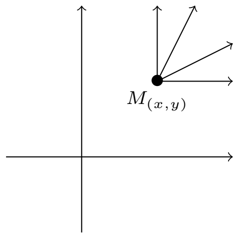
Example 2.4.1. Under Ψ, the morphisms $V_{(-1,\frac{1}{2}]} \to V_{(-1,1]} \to V_{(-1,2]}$ correspond to the morphisms $M_{(0,\ln\frac{1}{2})} \to M_{(0,0)} \to M_{(0,\ln 2)}$, graphically



The morphisms $V_{(-2,1]} \to V_{(-1,1]} \to V_{(-\frac{1}{2},1)}$ correspond to the morphisms $M_{(-\ln 2,0)} \to M_{(0,0)} \to M_{(\ln 2,0)}$, graphically



The support of the functor $\mathrm{Hom}_{\mathbb{R}^2}(M_{(x,y)}, -)$ is the northeast part of $M_{(x,y)}$ on the plane which looks like



2.5. **Certain Frobenius subcategories of $\mathbb{R}^2$, denoted by $\mathcal{F}_r$.** Let $r \in \mathbb{R}_{>0}$. The category $\mathcal{F}_r$ is defined to be the full subcategory of $\mathbb{R}^2$ with indecomposable objects $\{M_{(x,y)} \mid |x-y| \leq r\}$.

Definition 2.5.1. An *exact structure* on any additive category $\mathcal{F}$ is given by:
A collection of exact sequences $\mathcal{E} = \{0 \to A \to B \to C \to 0\}$ satisfying the following:

- $0 \to 0 \to 0 \to 0 \to 0 \in \mathcal{E}$.
- $\mathcal{E}$ is closed under pull-backs, i.e. suppose $0 \to A \to B \to C \to 0 \in \mathcal{E}$ and $f : X \to C$ is a morphism in $\mathcal{F}$, then there exists an exact sequence $0 \to A \to E \to X \to 0 \in \mathcal{E}$, such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

- $\mathcal{E}$ is closed under push-outs, i.e. suppose $0 \to A \to B \to C \to 0 \in \mathcal{E}$ and $f : A \to X$ is a morphism in $\mathcal{F}$, then there exists an exact sequence $0 \to X \to E \to C \to 0 \in \mathcal{E}$, such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

All the morphisms g such that $0 \to A \to B \xrightarrow{g} C \to 0 \in \mathcal{E}$ are defined to be *epimorphisms* and all the morphisms f such that $0 \to A \xrightarrow{f} B \to C \to 0 \in \mathcal{E}$ are defined to be *monomorphisms*.

- Epimorphisms are closed under compositions.
- Monomorphisms are closed under compositions.

Remark 2.5.2. (a) In the definition of exact structure, it suffices to assume that epimorphisms are closed under compositions or monomorphisms are closed under compositions.
(b) An additive category with an exact structure is called an *exact category*.

Proposition 2.5.3. *For each $r \in \mathbb{R}$ the category $\mathcal{F}_r$ is an exact category where the collection $\mathcal{E}$ consists of special exact sequences:*

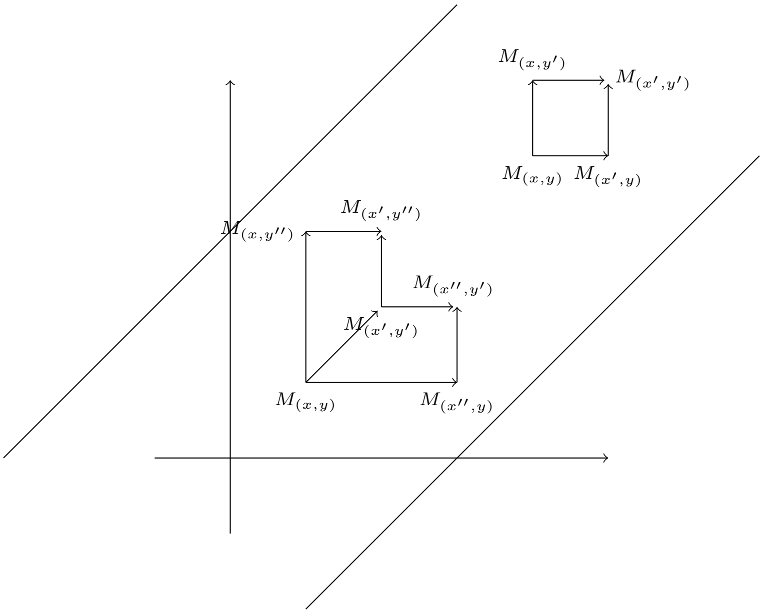
$$0 \longrightarrow \oplus_{i\in I} M_{(x_i,y_i)} \longrightarrow \oplus_{j\in J} M_{(x'_j,y'_j)} \longrightarrow \oplus_{t\in T} M_{(x''_t,y''_t)} \longrightarrow 0$$

satisfying $\{x_i\}_{i\in I} \cup \{x''_t\}_{t\in T} = \{x'_j\}_{j\in J}$ *and* $\{y_i\}_{i\in I} \cup \{y''_t\}_{t\in T} = \{y'_j\}_{j\in J}$.

Example 2.5.4. We graphically show two examples of typical exact sequences in $\mathcal{F}_r$:

$$0 \longrightarrow M_{(x,y)} \longrightarrow M_{(x',y)} \oplus M_{(x,y')} \longrightarrow M_{(x',y')} \longrightarrow 0,$$

$$0 \longrightarrow M_{(x,y)} \longrightarrow M_{(x,y'')} \oplus M_{(x',y')} \oplus M_{(x'',y)} \longrightarrow M_{(x',y'')} \oplus M_{(x'',y')} \longrightarrow 0.$$



Recall that the *projective objects* in an exact category are P such that for any exact sequence $0 \to A \to B \to C \to 0 \in \mathcal{E}$, any morphism $f : P \to C$ factors through B. The injective objects are defined dually.

Proposition 2.5.5. $\{M_{(x,x+r)}, M_{(x,x-r)}\}$ *are indecomposable projective-injective objects in $\mathcal{F}_r$ and all indecomposable projective-injective objects in $\mathcal{F}_r$ are of this form.*

Definition 2.5.6. An exact category $\mathcal{X}$ is called Frobenius if
- $\mathcal{X}$ has enough projectives (i.e. for any object X, there is an epimorphism $P \to X$ with P projective.)
- $\mathcal{X}$ has enough injectives (i.e. for any object X, there is a monomorphism $X \to I$ with I injective.)
- An object in $\mathcal{X}$ is projective if and only if it is injective.

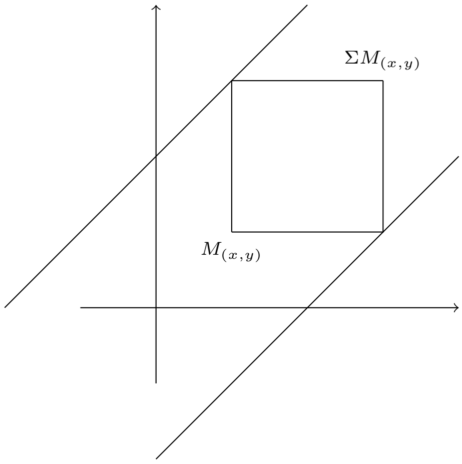
Theorem 2.5.7. *The exact category $\mathcal{F}_r$ is a Frobenius category.*

Additionally, $\mathcal{F}_r$ has the following property:
- For any object $X \in \mathcal{F}_r$, there is a projective cover $P \to X$.
- For any object $X \in \mathcal{F}_r$, there is an injective envelop $X \to I$.
- For each indecomposable object $X \in \mathcal{F}_r$, there is an exact sequence:

$$0 \to X \to Q \to \Sigma X \to 0,$$

where Q is a projective-injective object and ΣX is again indecomposable.

Example 2.5.8. By definition, it is easy to compute $\Sigma M_{(x,y)} \cong M_{(y+r,x+r)}$, graphically:

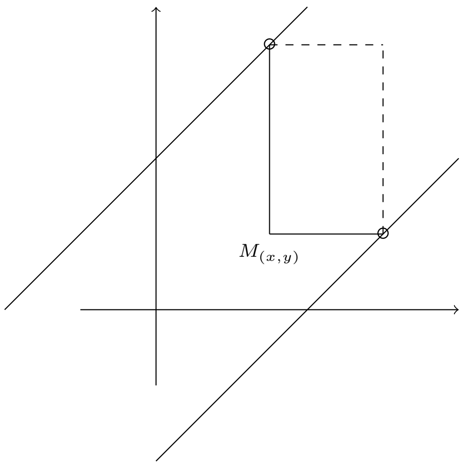


2.6. **Continuous derived category $\mathcal{D}_r$ as the stable category of $\mathcal{F}_r$.** Define the stable category $\underline{\mathcal{F}_r}$ as follows: $Obj(\underline{\mathcal{F}_r}) = Obj(\mathcal{F}_r)$. The morphisms are defined by

$$\mathrm{Hom}_{\underline{\mathcal{F}_r}}(X,Y) := \mathrm{Hom}_{\mathcal{F}_r}(X,Y)/P(X,Y),$$

where $P(X,Y)$ is the subgroup of morphisms $X \to Y$ which factor through any projective object in $\mathcal{F}_r$.

Example 2.6.1. We show an example of $Supp\,\mathrm{Hom}(M_{(x,y)},-)$ in the stable category $\underline{\mathcal{F}_r}$.



From the picture one can see that $\underline{\mathcal{F}_r}$ does not have Serre duality, and hence no Auslander Reiten translation. (The existence of Serre functor suggests $\mathrm{Hom}_{\underline{\mathcal{F}_r}}(X,-) \cong D\,\mathrm{Hom}_{\underline{\mathcal{F}_r}}(-,SX)$, but the support of any representable functor $\mathrm{Hom}_{\underline{\mathcal{F}_r}}(X,-)$ is never the same as the support of any representable functor $\mathrm{Hom}_{\underline{\mathcal{F}_r}}(-,Y)$.)

2.7. **Triangulated structure of $\mathcal{D}_r$.**

Theorem 2.7.1. [H] *Let $\mathcal{F}$ be a Frobenius category. Then the stable category $\underline{\mathcal{F}}$ is a triangulated category.*

Recall that $\mathcal{T}$ is a triangulated category if:
- There exists a natural equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$.
- There exists a collection of triangles $\mathcal{E}$ satisfying several axioms:

$(TR1)$ (i) $\mathcal{E}$ is closed under isomorphism. (ii) For any morphism $f : X \to Y$, there exists a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{E}$. (iii) $X \xrightarrow{=} X \to 0 \to \Sigma X \in \mathcal{E}$.
$(TR2)$ If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle, then so is $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.
$(TR3)$ For two triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ in $\mathcal{E}$. If there are morphisms u and v such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y', \end{array}$$

then there is a morphism w such that the following diagram is commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'. \end{array}$$

$(TR4)$ (Octahedral axiom) ... This is too long to state it here.

Proposition 2.7.2. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a triangle in a triangulated category $\mathcal{T}$, then for any object U there is a long exact sequence:*

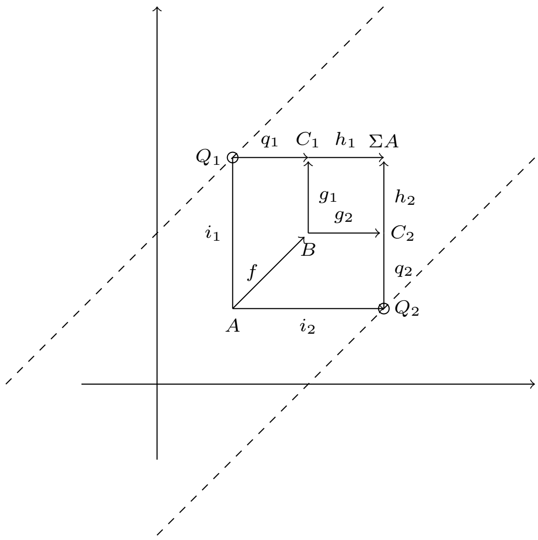
$$\cdots \to \mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y) \to \mathrm{Hom}(U, Z) \to \mathrm{Hom}(U, \Sigma X) \to \mathrm{Hom}(U, \Sigma Y) \to \cdots .$$

Also recall that in a Frobenius category $\mathcal{F}$, for each morphism $f : A \to B$, one can take the push out with the injective envelope $i : A \to Q$ and obtain the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i} & Q & \xrightarrow{p} & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q} & & \| \\ B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

Then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is a distinguished triangle in the stable category $\underline{\mathcal{F}}$ and all the triangles in $\underline{\mathcal{F}}$ is isomorphic to such triangles.

Example 2.7.3. Here is a typical example of a triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\underline{\mathcal{F}_r}$.

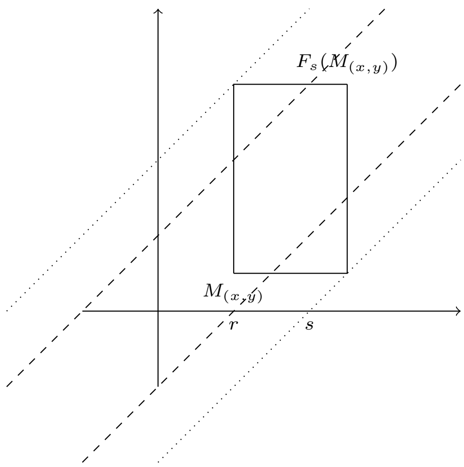


Remark 2.7.4. Because of similarities with the derived categories for acyclic quivers, we denote the stable category $\underline{\mathcal{F}_r}$ by $\mathcal{D}_r$ and sometimes call it *continuous derived category.*

2.8. **Orbit categories of $\mathcal{D}_r$, denoted by $\mathcal{O}_{r,s}$.** For each number $s \in \mathbb{R}_{>0}$, we can define a functor F_s on indecomposable objects as follows:

$$F_s(M_{(x,y)}) := M_{(y+s,x+s)}.$$

Graphically,



Definition 2.8.1. Define the orbit category $\mathcal{O}_{r,s} = \underline{\mathcal{F}_r}/F_s$ as follows:
The objects of $\mathcal{O}_{r,s}$ are the orbits of objects in $\underline{\mathcal{F}_r}$. Denote the orbits of $M_{(x,y)}$ by $\widetilde{M_{(x,y)}}$.

The morphisms are defined by $\mathrm{Hom}_{\mathcal{O}_{r,s}}(\tilde{M}_{(x,y)}, \tilde{M}_{(x',y')}) = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\underline{\mathcal{F}_r}}(M_{(x,y)}, F_s^i M_{(x',y')})$.
Notice that $\coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\underline{\mathcal{F}_r}}(M_{(x,y)}, F_s^i M_{(x',y')})$ is a finite sum for fixed r and s.

2.9. **When is $\mathcal{O}_{r,s}$ triangulated?**

Theorem 2.9.1. [IT15] *The orbit category $\mathcal{O}_{r,s}$ is a triangulated category if and only if $s \geq r$.*

2.10. **When does $\mathcal{O}_{r,s}$ have a cluster structure?** i.e. for which r and s is there a cluster structure in the sense of Buan, Iyama, Reiten, Scott paper [BIRS]?

Definition 2.10.1. [BIRS] A *cluster structure* on a triangulated category $\mathcal{X}$ is a collection of subcategories $\{\mathcal{T}\}$ satisfying:

(1) Given any indecomposable object $T_i \in \mathcal{T}$, there exists a unique indecomposalbe object T_i^* such that $add(ind\mathcal{T}\backslash\{T_i\} \cup \{T_i^*\}) \in \mathcal{T}$; replacing $\mathcal{T}$ by $\mathcal{T}\backslash\{T_i\} \cup \{T_i^*\}$ is called *mutation of $\mathcal{T}$ at T_i.*

(2) The objects T_i and T_i^* are related by triangles:
$$T_i^* \xrightarrow{g} B \xrightarrow{f} T_i \to T_i^*[1] \quad \text{and} \quad T_i \xrightarrow{g'} B' \xrightarrow{f'} T_i^* \to T_i[1]$$
where f and f' are right $\mathrm{add}(\mathcal{T}\backslash T_i)$-approximations of T_i and T_i^* (respectively), and g and g' are left $\mathrm{add}(\mathcal{T}\backslash T_i)$-approximation of T_i^* and T_i (respectively).

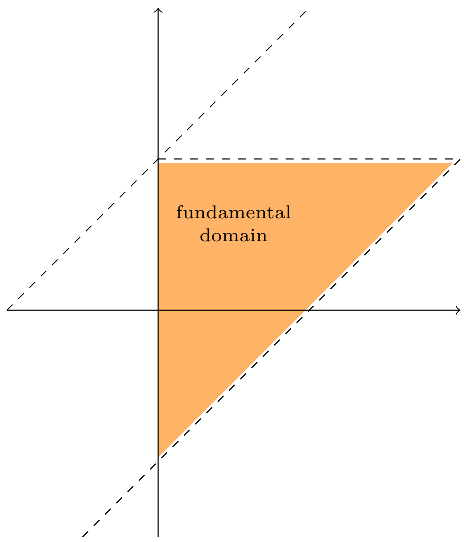
(3) The quivers of categories $\mathcal{T}$ do not contain any 1- or 2-cycles.
(4) The quivers of categories $\mathcal{T}$ and $\mu_i(\mathcal{T}) = (\mathcal{T}\backslash T_i) \cup T_i^*$ are related according to Fomin-Zelevinsky mutations.

Theorem 2.10.2. [IT15] *Let $r, s \in \mathbb{R}_{>0}$. The orbit category $\mathcal{O}_{r,s}$ has a cluster structure if and only if either $s = r$ or $\frac{r}{s} = \frac{n+1}{n+3}$ for some $n \in \mathbb{Z}_{>0}$.*

Definition 2.10.3. When $r = s$, denote the orbit category $\underline{\mathcal{F}_r}/F_s = \mathcal{O}_{r,s} =: \mathcal{C}_r$. This category is called *continuous cluster category.*

Theorem 2.10.4. [IT15] *If $\frac{r}{s} = \frac{n+1}{n+3}$ then the cluster category of $\mathbb{A}_n$ is embedded in $\mathcal{O}_{r,s}$ as a triangulated subcategory.*

2.11. **Cluster tilting subcategories in the continuous cluster category $\mathcal{C}_r$.** By definition $\mathcal{C}_r = \mathcal{O}_{r,r} = \underline{\mathcal{F}_r}/F_r$. A fundamental domain of $\mathcal{C}_r$ can be chosen as a triangle area:

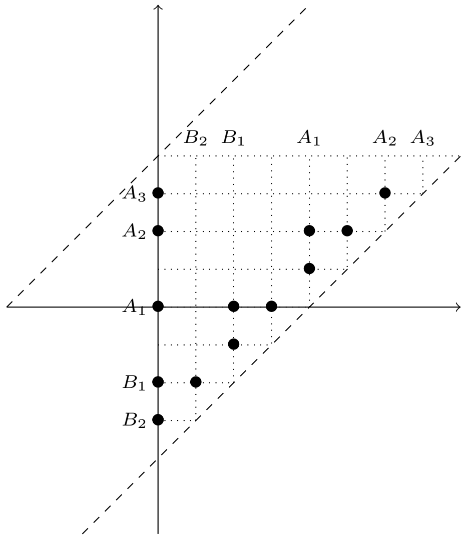


We now describe a collection of subcategories $\{\mathcal{T}\}$ which form a cluster structure on the orbit category $\mathcal{O}_{r,r}$, which was a motivation for the term "continuous cluster category". This was used in the proof of the Theorem 2.10.2.

Proposition 2.11.1. *Let $\mathcal{O}_{r,r}$ be the orbit category. Let $\{\mathcal{T}\}$ be the collection of all subcategories $\mathcal{T}$ of $\mathcal{O}_{r,r}$ where each $\mathcal{T}$ satisfies the following properties:*

(1) *For any two non-isomorphic T_i, T_j in $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{C}_r}(T_i,T_j)\oplus\mathrm{Hom}_{\mathcal{C}_r}(T_j,T_i)=\mathbf{k}$ or 0,*
(2) *The set of all representatives of all indecomposable objects of $\mathcal{T}$ forms a discrete subset of points in $\mathbb{R}^2$,*
(3) *The subcategory $\mathcal{T}$ is maximal with respect to the above properties.*

Example 2.11.2. This is an example of some objects in a cluster tilting subcategory $\mathcal{T}$ in $\mathcal{C}_r$. Some objects are indicated with their representatives in the fundamental domain, and also with another representative of the same object, outside of the fundamental domain.
The chosen objects all satisfy conditions (1) and (2).
In order to satisfy condition (3), infinitely many objects must be included, i.e. $\mathcal{T}$ contains infinitely many indecomposable objects (the picture only lists first a few of them).



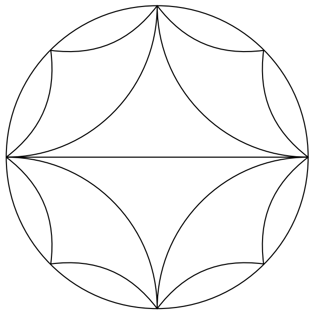
Remark 2.11.3. The continuous cluster category $\mathcal{C}_r$, in some sense, is quite different from the cluster categories of acyclic quivers:

(1) The categories $\mathcal{C}_r$ and $\mathcal{C}_{r'}$ are isomorphic for all $r,r'\in\mathbb{R}_{>0}$.
(2) $X\cong\Sigma X$ for all objects X in $\mathcal{C}_r$.
(3) There is no Serre functor on $\mathcal{C}_r$. Reason: from the above description of the fundamental domain, the support of $\mathrm{Hom}_{\mathcal{C}_r}(A_1,\_)$ is the square starting at A_1 which includes bottom and left boundaries but does not include top and right. However, support of each $\mathrm{Hom}_{\mathcal{C}_r}(\_\,,Y)$ is a square which includes top and right boundaries, but does not include left and bottom. Therefore fon any endofunctor S on $\mathcal{C}_r$, the support of $\mathrm{Hom}_{\mathcal{C}_r}(X,\_)$ can never be equal to the support of $\mathrm{Hom}_{\mathcal{C}_r}(\_\,,SX)$.
(4) There are no AR-triangles in $\mathcal{C}_r$. Reason: There are no right minimal almost split maps, since there are no right minimal maps of any kind.
(5) The continuous cluster category $\mathcal{C}_r$ is not 1-CY.
Reason: In general $\mathrm{Hom}_{\mathcal{C}_r}(X,Y)\ncong D\,\mathrm{Hom}_{\mathcal{C}_r}(Y,\Sigma X)=\mathrm{Hom}_{\mathcal{C}_r}(Y,X)$. For example $0\neq\mathrm{Hom}_{\mathcal{C}_r}(A_1,A_2)\ncong D\,\mathrm{Hom}_{\mathcal{C}_r}(A_2,A_1)=0$.
(6) $\mathcal{C}_r$ is not 2-CY. $\mathcal{C}_r$ is not n-CY for any n. Reason: the same as above.

2.12. **Ideal triangulations of the hyperbolic plane h.** Each cluster tilting subcategory $\mathcal{T}$ in the continuous cluster category $\mathcal{C}_r$ corresponds to a geodesic-triangulation of the hyperbolic plane **h** via the map:

$$\rho : \{\text{indecomposble objects in } \mathcal{C}_r\} \to \{\text{geodesics in } \mathbf{h}\}$$
$$M_{(x,y)} \mapsto (e^{(x/r)\pi i}, e^{((y/r)+1)\pi i}).$$

So the cluster tilting subcategory in the previous example corresponds to a triangulation which includes the following geodesics.



Remark 2.12.1. Let $\mathcal{C}_r$ be the continuous cluster category and $\{\mathcal{T}\}$ the cluster tilting structure as in the Proposition 2.11.1. Let ρ be the above map. Then:

(1) Each $\mathcal{T}$ corresponds to an ideal geodesic-triangulation of the hyperbolic plane.
(2) Each mutation of $\mathcal{T}$ in the direction of object T_i corresponds to Ptolemy mutation of the geodesic $\rho(T_i)$.
(3) All subcategories $\mathcal{T}$ are isomorphic.
(4) The group of orientation preserving homeomorphisms of the circle S^1 acts transitively on the cluster tilting subcategories $\{\mathcal{T}\}$.
(5) Mutation of a cluster tilting subcategory corresponds to group action of order 4.

3. Continuous cluster categories II

Certain stable categories of Frobenius categories of representations of the circle S^1 are shown to have cluster structure. [IT11]

3.1. **The representation of the circle S^1.** We define the representation of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ over $R = \mathbf{k}[[t]]$. Denote by $[x]$ the equivalent class of points on S^1 (under the equivalence $[x] = [x - 2\pi]$). Define a representation V as follows:

At each point, $V([x])$ is an R-module. For any $0 \le \alpha \le 2\pi$, the map $V^{(x,\alpha)} : V([x]) \to V([x-\alpha])$ is an R-homomorphism, $V^{(x,0)} : V([x]) \to V([x])$ is the identity map and the map $V^{(x,2\pi)} : V([x]) \to V([x-2\pi])(= V([x]))$ is the multiplication by t, such that $V^{(x,\alpha+\beta)} = V^{(x-\alpha,\beta)} \circ V^{(x,\alpha)}$.

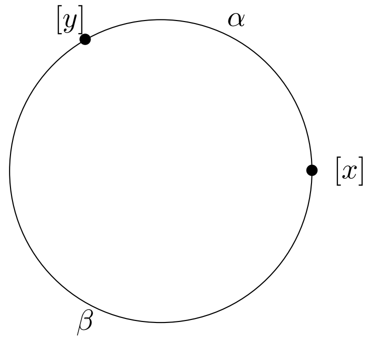
Suppose V and W are two representations of S^1, then a morphism $f : V \to W$ is defined as a collection of R-homomorphisms $\{f_{[x]}\}_{[x]\in S^1}$, such that for all $\alpha \ge 0$ the following diagram commutes:

$$\begin{array}{ccc} V([x]) & \xrightarrow{f_{[x]}} & W([x]) \\ \downarrow{\scriptstyle V^{(x,\alpha)}} & & \downarrow{\scriptstyle W^{(x,\alpha)}} \\ V([x-\alpha]) & \xrightarrow{f_{[x-\alpha]}} & W([x-\alpha]). \end{array}$$

The composition of morphisms are defined point-wise.

For each point $[x] \in S^1$, we define the (indecomposable) projective representation $P_{[x]}$ as: $P_{[x]}([y]) = R$ for all $[y] \in S^1$, $P_{[x]}^{(y,\alpha)} = 1_R$ for $0 \le \alpha < 2\pi$ and $P_{[x]}^{(y,2\pi)}$ is the multiplication by t.

3.2. **The Frobenius category $\mathcal{G}$.** The category $\mathcal{G}$ consists of indecomposable objects $E(x,y) = (P_{[x]} \oplus P_{[y]}, \varphi = \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix})$ where $\alpha : P_{[x]} \to P_{[y]}$ is the unique morphism which is the identity at $[x]$ and, $\beta : P_{[y]} \to P_{[x]}$ is the unique morphism which is the identity at $[y]$ and $\varphi = \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix} : P_{[x]} \oplus P_{[y]} \xrightarrow{\varphi} P_{[x]} \oplus P_{[y]}$ is a morphism induced by α and β, such that $\varphi^2 = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$.



A morphism $f : E(x,y) \to E(x',y')$ is a morphism $f : P_{[x]} \oplus P_{[y]} \to P_{[x']} \oplus P_{[y']}$ such that the following diagram is commutative:

$$\begin{array}{ccc} P_{[x]} \oplus P_{[y]} & \xrightarrow{\varphi} & P_{[x]} \oplus P_{[y]} \\ \downarrow f & & \downarrow f \\ P_{[x']} \oplus P_{[y']} & \xrightarrow{\varphi'} & P_{[x']} \oplus P_{[y']}. \end{array}$$

Theorem 3.2.1. [IT11] *$\mathcal{G}$ is a Frobenius category.*

In fact, the projective-injective objects are $E(x,x) = (P_{[x]} \oplus P_{[x]}, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix})$. In $\mathcal{G}$, for each indecomposable object A, there is a sequence $A \xrightarrow{i} Q \xrightarrow{p} \Sigma A$, where i is the injective envelope and p is the projective cover which can be described in the following way:

$$\begin{array}{ccccc} E(x,y) & \longrightarrow & E(y,y) \oplus E(x, x+2\pi) & \longrightarrow & E(y, x+2\pi) \\ \| & & \| & & \| \\ (P_{[x]} \oplus P_{[y]}, \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix}) & & (P_{[y]} \oplus P_{[y]}, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}) \oplus (P_{[x]} \oplus P_{[x+2\pi]}, \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}) & & (P_{[y]} \oplus P_{[x+2\pi]}, \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}) \end{array}$$

3.3. **The stable category of $\mathcal{G}$.** Let $\underline{\mathcal{G}}$ be the stable category of $\mathcal{G}$, then $\underline{\mathcal{G}}$ is a triangulated category by Happel's theorem.

Theorem 3.3.1. [IT11] *The stable category of $\mathcal{G}$ is isomorphic to the continuous cluster category, i.e. $\underline{\mathcal{G}} \cong \mathcal{C}_r$ as triangulated categories.*

Remark 3.3.2. With the above theorem, all the remarks about cluster tilting structure and ideal triangulations of the hyperbolic plane hold for the cluster structure of $\underline{\mathcal{G}}$. The cluster tilting subcategories in $\underline{\mathcal{G}}$ corresponds to ideal triangulations of the hyperbolic plane. Mutations in cluster tilting subcategories $\mathcal{T}$ correspond to Ptolemy mutations in the triangulation of the hyperbolic plane. All cluster tilting subcategories are isomorphic and the isomorphisms can be viewed as homeomorphisms of S^1. However, not all cluster tilting subcategories are in the same mutation class.

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